
HOMEWORK 4 (MODERN OPTIMIZATION ALGORITHMS, MATH 485)

Due date: See class webpage on Canvas.

Unless otherwise stated, $\mathcal{H}$ is a real Hilbert space with inner product $\langle \cdot | \cdot \rangle$ and induced norm $\| \cdot \|$.

f is *strictly convex* if

$$(\forall \alpha \in]0, 1[)(\forall x, y \in \text{dom } f \text{ s.t. } x \neq y) \quad f(\alpha x + (1 - \alpha)y) < \alpha f(x) + (1 - \alpha)f(y). \quad (1)$$

QUESTIONS

1. Suppose that $f \in \Gamma_0(\mathcal{H})$ is strictly convex and suppose that at least one minimizer exists, i.e., $\text{argmin}_{x \in \mathcal{H}} f(x) \neq \emptyset$. Show that the minimizer is unique.
2. Let $f, g \in \Gamma_0(\mathcal{H})$ satisfy $\text{dom } f \cap \text{dom } g \neq \emptyset$. Suppose that f is strictly convex. Show that $f + g$ is strictly convex.
3. This exercise demonstrates that the “usual” differentiation rules can change if you re-norm your Hilbert space. Consider the real Hilbert space $\mathcal{H} = \mathbb{R}^2$ equipped with the inner product

$$\langle (x_1, x_2) | (y_1, y_2) \rangle_{\mathcal{H}} = \frac{1}{4}x_1y_1 + \frac{3}{4}x_2y_2. \quad (2)$$

Note that $\langle \cdot | \cdot \rangle$ is not the standard dot product; please take a few minutes to review the inner product properties and verify that $\langle \cdot | \cdot \rangle$ is indeed an inner product¹. You don't have to submit anything related to this verification.

Consider the operation $f: \mathcal{H} \rightarrow \mathbb{R}: (x_1, x_2) \mapsto x_1^2 + x_2^2$. Compute ∇f and discuss how this differs from the standard power rule from introductory Calculus.

4. **EC:** This exercise demonstrates that, if one can show that function values converge to the optimal function value, this does not guarantee the iterates get anywhere close to the solution of your underlying problem.

Provide a function $f \in \Gamma_0(\mathcal{H})$ and a sequence $(x_n)_{n \in \mathbb{N}}$ such that all three properties hold.

- $\text{Argmin } f \neq \emptyset$ and
- The function values approach optimality: $\lim_{n \rightarrow \infty} f(x_n) = \min_{x \in \mathcal{H}} f(x)$
- The minimizers do not approach $\text{Argmin } f$: For every $z \in \text{Argmin } f$, $\|x - z\| \not\rightarrow 0$; equivalently, there exists a fixed constant $B > 0$ such that $\text{Argmin } f(x_n) \geq B$.

¹hint: verifying this relies on the fact that $\frac{1}{4} + \frac{3}{4} = 1$