

Signal recovery from inconsistent nonlinear observations

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Joint work with Patrick L. Combettes (NC State University)

RT MIA workshop on **Imaging inverse problems - regularization, low dimensional models and applications**

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Outline

- Setting and history
- **Firmly nonexpansive** equations
- Feasibility problems involving such equations
 - relaxation for inconsistent problems
 - “regularization”
 - Theory & numerics

Motivation: the linear setting

Youla's Model, 1978

Let U_1 and U_2 be closed vector subspaces of a real Hilbert space \mathcal{H} . Given $p \in U_2$,

find $x \in U_1$ such that $\text{proj}_{U_2} x = p$.

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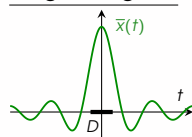
Example: Bandlimited extrapolation (Papoulis, 1975)

Let $\sigma > 0$, $D \subset \mathbb{R}$, and $p = \bar{x}|_D$.

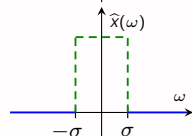
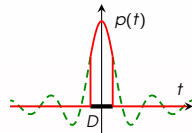
Goal: find x such that

$$\begin{cases} p = x|_D \text{ a.e.} \\ \hat{x} = 0 \text{ outside of } [-\sigma, \sigma] \text{ a.e.} \end{cases}$$

Original signal:



Given info:



Extension of the linear setting

Combettes & Reyes, 2010

Let K be a finite set. For every $k \in K$, let U_k be a closed vector subspace of \mathcal{H} , and let $p_k \in U_k$. The goal is to

find $x \in \mathcal{H}$ such that $(\forall k \in K) \text{proj}_{U_k} x = p_k$.

- Projection methods are available for finding solutions.
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However, there are many applications in which we seek to solve

$$(\forall k \in K) \quad F_k x = p_k,$$

where $(F_k)_{k \in K}$ are **nonlinear** operators on a real Hilbert space \mathcal{H} .

Our setting

Let \mathcal{H} be a real Hilbert space. The operator $F: \mathcal{H} \rightarrow \mathcal{H}$ is **firmly nonexpansive** if

$$(\forall (x, y) \in \mathcal{H}^2) \quad \|Fx - Fy\|^2 \leq \|x - y\|^2 - \|(\text{Id} - F)x - (\text{Id} - F)y\|^2.$$

- General enough to capture many applications.
- Sufficiently structured to yield tractable, efficient algorithms which converge to a solution from any initial point.
- Special case: Proximity operators (e.g., Projections onto closed convex sets.)

Roadblocks

Let $F: \mathcal{H} \rightarrow \mathcal{H}$ be firmly nonexpansive.

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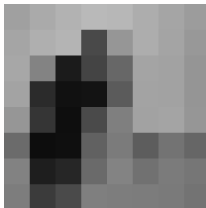
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Difficulties:

- $\|F(\cdot) - p\|$ is typically **nonconvex**.
 - Convex minimization tools cannot be used.
 - Guarantees of convergence to a solution are rare.
- In general, **projecting** onto $F^{-1}(\{p\})$ is **not possible**.
 - Cannot be solved using **projection methods**.

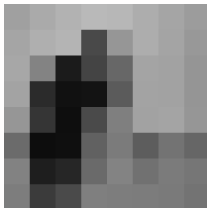
Examples: projections

- Dimension reduction and saturation



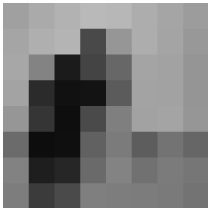
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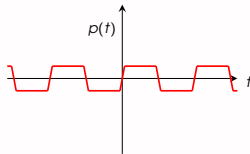
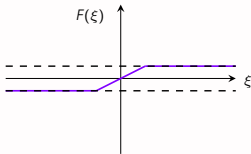
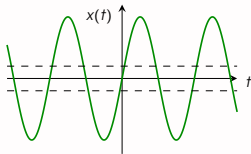


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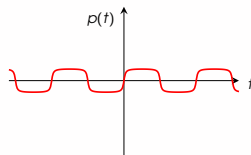
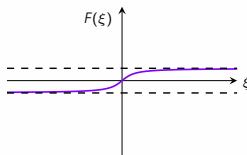
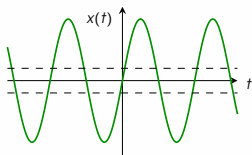


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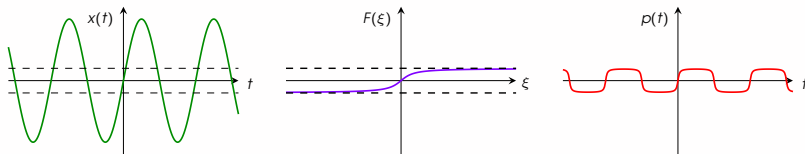
Examples

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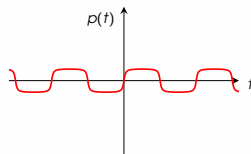
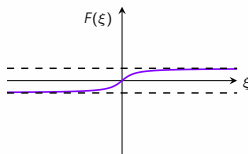
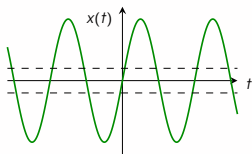
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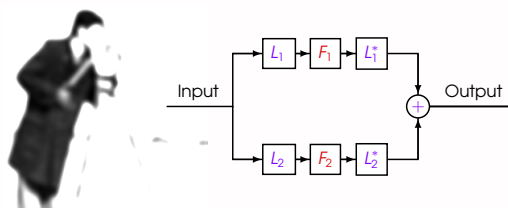
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Examples: “proxification”

Definition: Given $Q: \mathcal{H} \rightarrow \mathcal{H}$ and $q \in \text{ran} Q$, (Q, q) is **proxifiable** if there exists $F: \mathcal{H} \rightarrow \mathcal{H}$ which is **firmly nonexpansive** and $p \in \text{ran} F$ such that

$$(\forall x \in \mathcal{H}) \quad Qx = q \quad \Leftrightarrow \quad Fx = p$$

(1)

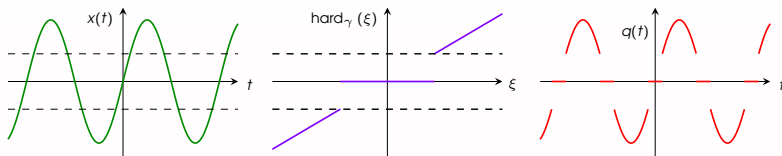
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Example: Hard thresholding at level $\gamma > 0$

$$\text{hard}_\gamma : \xi \mapsto \begin{cases} \xi, & \text{if } |\xi| > \gamma; \\ 0, & \text{if } |\xi| \leq \gamma, \end{cases} \quad (1)$$



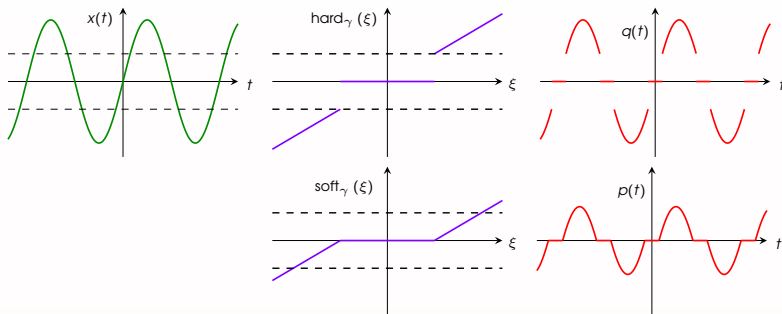
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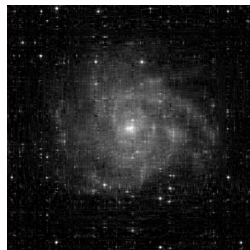
Let $\mathcal{H} = \mathbb{R}^{N \times M}$, set $s = \min\{N, M\}$, let $\gamma > 0$, and denote the singular value decomposition of $x \in \mathcal{H}$ by

$$x = U_x \operatorname{diag}(\sigma_1(x), \dots, \sigma_s(x)) V_x^\top. \quad (2)$$

A **low rank approximation** q of x is

$$U_x \operatorname{diag}\left(\operatorname{hard}_\gamma(\sigma_1(x)), \dots, \operatorname{hard}_\gamma(\sigma_s(x))\right) V_x^\top. \quad (3)$$

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$F: \mathcal{H} \rightarrow \mathcal{H}: x \mapsto$

$U_x \operatorname{diag}(\operatorname{soft}_\gamma(\sigma_1(x)), \dots, \operatorname{soft}_\gamma(\sigma_s(x))) V_x^\top,$

and construct p by shifting the nonzero singular values of q by $-\gamma$.



Feasibility

We seek to recover a signal \bar{x} in a real Hilbert space \mathcal{H} from

- A finite number of **transformations** $(p_k)_{k \in K}$ of the form

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Problem 1

$$\text{find } x \in \underbrace{\bigcap_{j \in J} C_j}_{\text{Prior information}} \text{ such that } (\forall k \in K) \underbrace{F_k x = p_k}_{\text{Transformations}},$$

assuming at least one solution exists.

Foregoing minimization: directly to fixed points.

Main ingredients:

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- Algorithm and numerics:



P. L. Combettes and ZCW, [A fixed point framework for recovering signals from nonlinear transformations](#), 2020 Proc. Eur. Signal Process. Soc., pp. 2120–2124. Amsterdam, The Netherlands, Jan. 18–22, 2021.

Inconsistent feasibility

Let $C \subset \mathcal{H}$ be nonempty closed and convex and let I be finite. For every $i \in I$, let \mathcal{G}_i be a real Hilbert space, let $p_i \in \mathcal{G}_i$, let $L_i: \mathcal{H} \rightarrow \mathcal{G}_i$ be a nonzero bounded linear operator, and let $F_i: \mathcal{G}_i \rightarrow \mathcal{G}_i$ be a firmly nonexpansive operator. The goal is to

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Problem 3: A variational inequality relaxation of (4)

Let $(\omega_i)_{i \in I}$ be real numbers in $]0, 1]$ such that $\sum_{i \in I} \omega_i = 1$.

$$\text{find } x \in C \text{ such that } (\forall y \in C) \quad \sum_{i \in I} \omega_i \langle L_i(y - x) \mid F_i(L_i x) - p_i \rangle \geq 0.$$

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- If (4) has a solution, then it is equivalent to Problem 3.
- Problem 3 is guaranteed to possess solutions under mild conditions.

Intuition: relaxed problem

Example 1 of Problem 3

Let $\beta > 0$ and let $f: \mathcal{H} \rightarrow \mathbb{R}$ be convex with a β^{-1} -Lipschitzian gradient. Set $F_1 = \beta \nabla f$, $p_1 = 0$, and $L_1 = \text{Id}$. Then (4) is equivalent to

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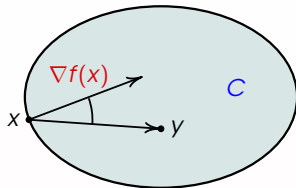
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and Problem 3 is equivalent to

find $x \in C$ such that $(\forall y \in C) \langle y - x \mid \nabla f(x) \rangle \geq 0$,

i.e.,
$$\underset{x \in C}{\text{minimize}} \quad f(x).$$



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Existence of solutions and a **block-iterative** algorithm for finding them:



P. L. Combettes and ZCW, **A variational inequality model for the construction of signals from inconsistent nonlinear equations**,

SIAM J. Imaging Sci., vol. 15, no. 1, pp. 84–109, 2022.

Existence results

Notation: N_C is the **normal cone** operator of C .

Proposition

Problem 3 admits a solution in each of the following instances.

- ❶ $\sum_{i \in I} \omega_i L_i^* p_i \in \text{ran}(N_C + \sum_{i \in I} \omega_i L_i^* \circ F_i \circ L_i)$.
- ❷ C is bounded.
- ❸ $\text{ran} N_C + \sum_{i \in I} \omega_i L_i^* (\text{ran} F_i) = \mathcal{H}$.
- ❹ For some $i \in I$, L_i^* is surjective and one of the following holds:
 - ❶ $L_i^* (\text{ran} F_i) = \mathcal{H}$.
 - ❷ F_i is surjective.
 - ❸ $\|F_i(y)\| \rightarrow +\infty$ as $\|y\| \rightarrow +\infty$.
 - ❹ $\text{ran}(Id - F_i)$ is bounded.
 - ❺ There exists a continuous convex function $g_i: \mathcal{G}_i \rightarrow \mathbb{R}$ such that $F_i = \text{prox}_{g_i}$.

Algorithm

Adapting an algorithm from



P. L. Combettes and L. E. Glaudin, [Solving composite fixed point problems with block updates](#)

Adv. Nonlinear Anal.,
vol. 10, pp. 1154–1177,
2021.

we arrive at a
block-iterative solution
method.

Let $x_0 \in \mathcal{H}$, let $\gamma \in]0, 2[$, and, for every $i \in I$, let $t_{i,-1} \in \mathcal{H}$ and set $\gamma_i = \gamma / \|L_i\|^2$. Iterate

for $n = 0, 1, \dots$

$\left[\begin{array}{l} \emptyset \neq I_n \subset I \\ \text{for every } i \in I_n \\ \quad \left[\begin{array}{l} t_{i,n} = x_n - \gamma_i L_i^* (F_i(L_i x_n) - p_i) \end{array} \right. \\ \text{for every } i \in I \setminus I_n \\ \quad \left[\begin{array}{l} t_{i,n} = t_{i,n-1} \end{array} \right. \\ x_{n+1} = \text{proj}_C \left(\sum_{i=1}^m \omega_i t_{i,n} \right) \end{array} \right.$

Then under a mild condition on $(I_n)_{n \in \mathbb{N}}$, $(x_n)_{n \in \mathbb{N}}$ converges weakly to a solution to Problem 3.

Numerics: inconsistent image recovery

Experiment: $C = [0, 255]^N$ ($N = 256^2$), given noisy estimates of:

- Mean pixel value
- Fourier phase



Numerics: inconsistent image recovery

Experiment: $C = [0, 255]^N$ ($N = 256^2$), given noisy estimates of:

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- Fourier phase
- A blurred and saturated observation



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This problem is inconsistent.



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Numerics: promoting sparsity

Experiment: Given $C = [0, 255]^N$ ($N = 256$) and

- A low rank approximation.
- \bar{x} is sparse.

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Numerics: promoting sparsity

Experiment: Given $C = [0, 255]^N$ ($N = 256$) and

- A low rank approximation.
- \bar{x} is sparse. So, we set $\gamma = 1.5$,
 $F_2 = \text{Id} - \text{prox}_{\gamma \|\cdot\|_1} = \text{proj}_{B_\infty(0; \gamma)}$ and $p_2 = 0$.

Motivation:

$$F_2 x = p_2 \Leftrightarrow x \in \operatorname{argmin} \|\cdot\|_1.$$

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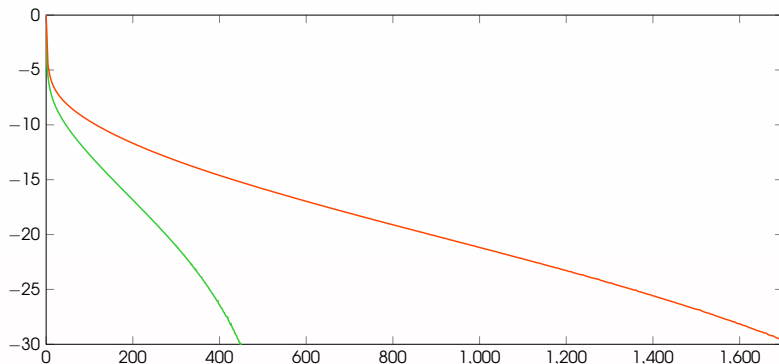
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Numerics: promoting sparsity

F_1 is expensive to compute.



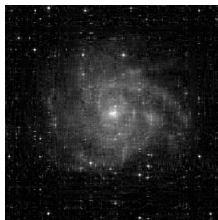
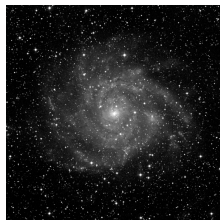
Relative error (dB) versus execution time (seconds) for **full-activation**, i.e., $l_n = I$ versus **block activation**, i.e.,

$$(\forall n \in \mathbb{N}) \quad l_n = \begin{cases} \{1, 2\}, & \text{if } n \equiv 0 \pmod{5}; \\ \{2\}, & \text{if } n \not\equiv 0 \pmod{5}. \end{cases}$$

Inconsistent feasibility

Goal: Separate the background of stars \bar{x}_1 from the galaxy \bar{x}_2 , given $C = [0, 255]^N$ ($N = 600^2$) and

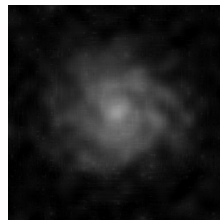
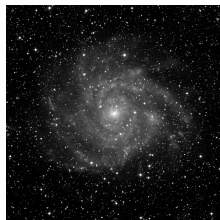
- A low rank approximation of the superposition $\bar{x}_1 + \bar{x}_2$
- \bar{x}_1 is sparse and \bar{x}_2 is sparse under the discrete cosine transform $L: \mathbb{R}^N \rightarrow \mathbb{R}^N$. We set $L_2: (x_1, x_2) \mapsto (x_1, Lx_2)$, $p_2 = 0$, and $F_2: (y_1, y_2) \mapsto (\text{proj}_{B_\infty(0;10)} y_1, \text{proj}_{B_\infty(0;45)} y_2)$.



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